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On recurrence equations associated with invariant varieties of periodic points

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Abstract

A recurrence equation is a discrete integrable equation whose solutions are all periodic and the period is fixed. We show that infinitely many recurrence equations can be derived from the information about invariant varieties of periodic points of higher dimensional integrable maps.

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1. Introduction

A recurrence equation is a discrete integrable equation whose solutions are all periodic and the period is fixed. Some of them had been known for some years, while others have been found recently. In this contribution we would like to show that infinitely many recurrence equations can be derived from the information about invariant varieties of periodic points of the higher dimensional integrable maps. Especially the recurrence equations associated with the Quispel, Roberts and Thompson (QRT) map [1] are shown to exist one for each period.

Some examples of the recurrence equations are [2]

$$x_{n+1} = \frac{a}{x_n}, \qquad a: \quad \text{constant} \tag{1}$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \tag{2}$$

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}}. \tag{3}$$

An interesting feature of these equations is that, for an arbitrary initial value, the solution is always periodic with a fixed period. The period is 2 in the case of (1), 5 in the case of (2) and 8 in the case of (3). These equations were named the recurrence equations by the authors of [3, 4] who have found many other examples of this type recently.

Apparently the recurrence equations are integrable. However there has not been known, to our knowledge, any method to find them systematically. The purpose of this paper is to develop a method to derive the recurrence equations from integrable maps in a systematic way.

Our key observation is that, writing (x_n, x_{n-1}, x_{n-2}) as (x, y, z), the recurrence equations (1)–(3) are equivalent to the higher dimensional maps

$$x \to X = \frac{a}{x} \tag{4}$$

$$(x, y) \to (X, Y) = \left(\frac{1+x}{y}, x\right) \tag{5}$$

$$(x, y, z) \to (X, Y, Z) = \left(\frac{1+x+y}{z}, x, y\right), \tag{6}$$

respectively. There are a pair of fixed points at $x = \pm\sqrt{a}, x = y = (1 \pm \sqrt{5})/2$ and $x = y = z = 1 \pm \sqrt{2}$ for each map (4)–(6). Otherwise an arbitrary point on the complex space can be an initial point of the periodic map of the corresponding period.

We have shown, in our recent paper [5, 6], that periodic points of higher dimensional integrable maps with some invariants form an invariant variety for each period. The invariant variety of periodic points is determined by imposing certain relations among the invariants. Every point on an invariant variety can be an initial point of the periodic map of the same period. All images of the map stay on this invariant variety. Therefore the map defines a recurrence equation of the fixed period if it is constrained on the invariant variety. In some cases the invariant varieties can be derived iteratively for all periods. We can associate one recurrence equation with every invariant variety, thus obtain infinitely many recurrence equations.

We explain briefly the notion of invariant varieties of periodic points in section 2. Many recurrence equations associated with the invariant varieties will be derived in section 3. We discuss, in section 4, a method which enables us to derive the series of recurrence equations.

2. Invariant varieties of periodic points

Let us consider an iteration of a rational map on $\hat{\mathbf{C}}^d$, where $\hat{\mathbf{C}} = \{\mathbf{C}, \infty\}$,

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \quad \to \quad \mathbf{X} = (X_1, X_2, \ldots, X_d) =: \mathbf{X}^{(1)}, \tag{7}$$

and assume that $H_1(\mathbf{x}), H_2(\mathbf{x}), \ldots, H_p(\mathbf{x})$ are the p invariants. We are interested in the behaviour of periodic points satisfying the conditions

$$\mathbf{X}^{(n)} = \mathbf{x}, \qquad n = 2, 3, \ldots. \tag{8}$$

If $h_1, h_2, \ldots, h_p$ are the values of the invariants determined by the initial point, then map (7) is constrained on the $(d-p)$-dimensional algebraic variety $V(h)$,

$$V(h) = \{\mathbf{x} | H_i(\mathbf{x}) = h_i, i = 1, 2, \ldots, p\}, \tag{9}$$

and the periodicity conditions (8) are reduced to the constraints on some $d-p$ functions $\Gamma_\alpha^{(n)}$,

$$\Gamma_\alpha^{(n)}(h_1, h_2, \ldots, h_p, \xi_1, \xi_2, \ldots, \xi_{d-p}) = 0, \qquad \alpha = 1, 2, \ldots, d-p, \qquad n \geqslant 2. \tag{10}$$

Here by $\xi_1, \xi_2, \ldots, \xi_{d-p}$ we denote the variables which parameterize the variety $V(h)$ after the elimination of the p components of $\mathbf{x}$. Note that the fixed-point conditions $(n = 1)$ are excluded in (10) since they have nothing to do with the invariants.

For an arbitrary set of values of $h_1, h_2, \ldots, h_p$, the functions $\Gamma_n^{(\alpha)}(h, \xi)$ define an affine variety, which we denote by $V^{(n)}(\langle\Gamma\rangle)$, i.e.,

$$V^{(n)}(\langle\Gamma\rangle) = \left\{\xi \,\middle|\, \Gamma_\alpha^{(n)}(h, \xi) = 0, \alpha = 1, 2, \ldots, d-p\right\}, \qquad n \geqslant 2.$$

In general this variety consists of a finite number of isolated points on $V(h)$, hence zero dimension, corresponding to the solutions to the $d-p$ algebraic equation (10) for the $d-p$ variables $\xi_1, \xi_2, \ldots, \xi_{d-p}$. In this case we say that the periodicity conditions (8) are 'uncorrelated'. If the values of the invariants are changed continuously these points move all together and form a subvariety of dimension p in $\hat{\mathbf{C}}^d$. Needless to say, this case includes a map with no invariant.

There are possibilities that equations (10) impose relations on $h_1, h_2, \ldots, h_p$ instead of fixing all ξ_α's. Let l be the number of such equations. We write them as

$$\gamma_\alpha^{(n)}(h_1, h_2, \ldots, h_p) = 0, \qquad \alpha = 1, 2, \ldots, l, \tag{11}$$

instead of $\Gamma_\alpha^{(n)}$, to emphasize independence from ξ_j's. If m is the number of the rest of the equations

$$\Gamma_\alpha^{(n)}(h_1, h_2, \ldots, h_p, \xi_1, \xi_2, \ldots, \xi_{d-p}) = 0, \qquad \alpha = 1, 2, \ldots, m \tag{12}$$

$d-p-m$ variables are not determined from the periodicity conditions. This means that $V^{(n)}(\langle\Gamma\rangle)$ forms a subvariety of dimension $d-p-m$ of $V(h)$. We say that the periodicity conditions are 'correlated' in this case. In [5] we have proved the following lemma:

Lemma [5]**.** *A set of correlated periodicity conditions satisfying* $\min\{p, d-p\} \geqslant l+m$ *and a set of uncorrelated periodicity conditions of a different period do not exist in one map simultaneously.*

When $m = 0$, in particular, the periodicity conditions determine none of the variables $\xi_1, \xi_2, \ldots, \xi_{d-p}$ but impose l relations among the invariants. Then the affine variety $V^{(n)}(\langle\Gamma\rangle)$ coincides with $V(h)$. In other words every point on $V(h)$ is a periodic point of period n, while $V(h)$ itself is constrained by the relations among the invariants. We say that the periodicity conditions are 'fully correlated' in this particular case. If we replace h_i by $H_i(\mathbf{x})$ in $\gamma_\alpha^{(n)}(h)$ the periodicity conditions (11) enable us to consider the constraints on the invariants as the constraints on the variables $\mathbf{x}$. We denote by $v^{(n)}(\langle\gamma\rangle)$ the affine variety generated by the functions $\gamma_\alpha^{(n)}(H_1(\mathbf{x}), H_2(\mathbf{x}), \ldots, H_p(\mathbf{x}))$, and distinguish it from $V^{(n)}(\langle\Gamma\rangle)$. Namely we define

$$v^{(n)}(\langle\gamma\rangle) = \left\{\mathbf{x} \,\middle|\, \gamma_\alpha^{(n)}(H_1(\mathbf{x}), H_2(\mathbf{x}), \ldots, H_p(\mathbf{x})) = 0, \alpha = 1, 2, \ldots, l\right\}. \tag{13}$$

We call $v^{(n)}(\langle\gamma\rangle)$ 'an invariant variety of periodic points', whose properties can be summarized as follows:

- The dimension of $v^{(n)}(\langle\gamma\rangle)$ is $d-l\ (\geqslant p)$.
- Every point on $v^{(n)}(\langle\gamma\rangle)$ can be an initial point of the periodic map of period n.
- All images of the periodic map starting from a point of $v^{(n)}(\langle\gamma\rangle)$ stay on it.
- $v^{(n)}(\langle\gamma\rangle)$ is determined by the invariants of the map alone.

If the periodicity conditions of period n are fully correlated, i.e., when $m = 0$ in (11), the condition $\min\{p, d-p\} \geqslant l+m$ is always satisfied as long as (11) has solutions. Our theorem thus follows from the lemma and this fact immediately.

Theorem [5]**.** *If there are an invariant variety of periodic points of some period, there is no set of isolated periodic points of other period in the map.*

This theorem tells us nothing about the integrability of a map. To proceed further we assume that a nonintegrable map has at least one set of uncorrelated periodicity conditions. This is certainly true if the map has a Julia set. Once we adopt this observation as a working hypothesis, our theorem is equivalent to the following statement:

If a map has an invariant variety of periodic points of some period, it is integrable.

In order to support this proposition, we have investigated various known integrable maps and found invariant varieties of periodic points in all cases if there are invariants.

3. Derivation of recurrence equations

If there are an invariant variety of periodic points of period n, every point on the variety can be an initial point of an n periodic map. All images of the map are on the variety before the map returns to the initial point. Therefore this variety is clearly distinguished from the rest of $\hat{\mathbf{C}}^d$ and is reserved only for the maps of period n. In other words, if the initial point is on this variety the map is always n period.

This fact enables us to derive a recurrence equation once an invariant variety of periodic points are known. Let

$$x_j \to X_j = f_j(x_1, x_2, \ldots, x_d), \qquad j = 1, 2, \ldots, d \tag{14}$$

be the map and (13) be the invariant variety of period n of this map. We solve

$$\gamma_\alpha^{(n)}(H_1(\mathbf{x}), H_2(\mathbf{x}), \ldots, H_p(\mathbf{x})) = 0, \qquad \alpha = 1, 2, \ldots, l \tag{15}$$

for l variables, say $x_{d-l+1}, \ldots, x_d$, and substitute them into f_j, $j = 1, 2, \ldots, d-l$ of (14). Then every initial point of the map $X_j = f_j$, $j = 1, 2, \ldots, d-l$, is constrained on $v^{(n)}(\langle\gamma\rangle)$, thus we obtain a recurrence equation of period n.

The simplest method to achieve this program is to find the lth elimination ideal of the functions $\{X_j - f_j, j = 1, 2, \ldots, d-l\}$ and $\left\{\gamma_\alpha^{(n)}, \alpha = 1, 2, \ldots, l\right\}$. If the ideal is generated by the functions $F_j^{(n)}$'s satisfying

$$F_j^{(n)}(X_1, X_2, \ldots, X_{d-l}, x_1, x_2, \ldots, x_{d-l}) = 0, \qquad j = 1, 2, \ldots, d-l, \tag{16}$$

the recurrence equations are obtained by solving (16) for $X_1, X_2, \ldots, X_{d-l}$.

For an illustration let us consider the map

$$(x, y) \to (X, Y) = \left(xy, \frac{y(1+x)}{1+xy}\right). \tag{17}$$

This map has one invariant $H(x, y) = y(1+x)$ and the invariant variety of period 3 is given by the zeros of

$$\begin{aligned}\gamma^{(3)}(x, y) &= H^2 + H + 1 \\ &= x^2y^2 + 2xy^2 + y^2 + xy + y + 1.\end{aligned} \tag{18}$$

The first elimination ideal of the functions $X - xy$ and (18) is generated by the function

$$F^{(3)}(X, x) = (x+1)^2X^2 + x(x+1)X + x^2,$$

from which we obtain two maps:

$$x \to X = \begin{cases} \omega \dfrac{x}{x+1}, \\ \omega^2 \dfrac{x}{x+1} \end{cases} \qquad (\omega^3 = 1). \tag{19}$$

The iteration of the first map yields

$$x \to \omega \frac{x}{x+1} \to \frac{\omega^2 x}{-\omega^2 x + 1} \to x,$$

while the second map yields

$$x \to \omega^2 \frac{x}{x+1} \to \frac{\omega x}{-\omega x + 1} \to x.$$

In the rest of this section we would like to present the various types of recurrence equations associated with invariant varieties of some integrable maps.

The d-dimensional Lotka–Volterra map is obtained by solving [7]

$$X_j(1 - X_{j-1}) = x_j(1 - x_{j+1}), \qquad j = 1, 2, \ldots, d \tag{20}$$

for $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ under the conditions $x_{j+d} = x_j (j = 1, 2, \ldots, d)$. The invariants of this map are given by [5]

$$\begin{cases} H_k = \sum'_{j_1, j_2, \ldots, j_k} x_{j_1} x_{j_2} \cdots x_{j_k} (1 - x_{j_1 - 1})(1 - x_{j_2 - 1}) \cdots (1 - x_{j_k - 1}) \\ \qquad\qquad\qquad (k = 1, 2, \ldots, [d/2]) \\ r = x_1 x_2 \cdots x_d \end{cases} \tag{21}$$

Here, the prime in the summation $\sum'$ of (21) means that the summation must be taken over all possible combinations $j_1, j_2, \ldots, j_k$ but excluding direct neighbours. The total number of the invariants is $p = [d/2] + 1$, where $[d/2] = d/2$ if d is even and $[d/2] = (d-1)/2$ if d is odd.

The invariant varieties have been derived in the cases of $d = 3$, 4 and 5 for some periods, explicitly [5]. In all examples the dimension of the invariant varieties is p. Hence the dimension of the recurrence equations is also p.

The three-dimensional Lotka–Volterra map is given by, writing $(x_1, x_2, x_3) = (x, y, z)$,

$$X = x\frac{1 - y + yz}{1 - z + zx}, \qquad Y = y\frac{1 - z + zx}{1 - x + xy}, \qquad Z = z\frac{1 - x + xy}{1 - y + yz}, \tag{22}$$

after solving (20) for X, Y, Z. There are two invariants

$$r = xyz, \qquad s = (1-x)(1-y)(1-z). \tag{23}$$

The invariant varieties of periodic points have dimension 2 and are generated by the functions:

$$\begin{aligned} \gamma^{(2)} &= s + 1 \\ \gamma^{(3)} &= r^2 + s^2 - rs + r + s + 1 \\ \gamma^{(4)} &= r^3 s + s^3 - 3rs^2 + 6r^2 s + 3rs - r^3 + s \\ \gamma^{(5)} &= r^3 s^4 - r^3 s^2 - 6r^4 s^5 + 10r^3 s^6 + 3s^5 r + s^6 + s^5 + 3r^4 s^4 - 3r^5 s^3 \\ &\quad - 6r^4 s^3 - r^6 s^3 + 3r^5 s^4 + s^4 + 21 s^4 r^2 + 6s^4 r + r^3 s^7 + s^7 \\ &\quad + 27 s^5 r^2 - 3s^6 r - r^3 s^5 + 21 r^2 s^6 - 10 r^3 s^3 - 6rs^7 + s^8 \\ &\;\;\vdots \end{aligned} \tag{24}$$

for the periods 2, 3, 4, 5, ..., respectively.

From these data we can derive a set of recurrence equations for each period. For the period 2 case we find

$$F_1^{(2)} = (x-1)X - x, \qquad F_2^{(2)} = (y-1)Y - y,$$

and the map is simply given as

$$(x, y) \to \left(\frac{x}{x-1}, \frac{y}{y-1}\right) \to (x, y). \tag{25}$$

In the period 3 case we obtain

$$\begin{aligned}F_1^{(3)} &= (x^2 - x + 1)X^2 + x(xy - 2x + y + 1)X + x^2(y^2 - y + 1)\\ F_2^{(3)} &= ((3x^2 - 3x + 1)y^2 - (3x^2 - 5x + 2)y + (x - 1)^2)Y^2\\ &\quad - y((3x^2 - 2x + 1)y - (2x - 1)(x - 1))Y + y^2(x^2 - x + 1).\end{aligned}$$

Since the solutions of $\left\{F_1^{(3)} = 0, F_2^{(3)} = 0\right\}$ are twofolds the map has two routes:

$$\begin{aligned}(x, y) &\to \left(\omega\frac{x(y + \omega^2)}{x + \omega^2}, \frac{(1 - \omega^2)y(x + \omega^2)}{3xy + (x + y - 1)\omega^2}\right)\\ &\to \left(\frac{(1 - \omega^2)x(y + \omega^2)}{3xy + (x + y - 1)\omega^2}, \omega\frac{y(x + \omega^2)}{y + \omega^2}\right) \to (x, y)\\ (x, y) &\to \left(\omega^2\frac{x(y + \omega)}{x + \omega}, \frac{(1 - \omega)y(x + \omega)}{3xy + (x + y - 1)\omega}\right)\\ &\to \left(\frac{(1 - \omega)x(y + \omega)}{3xy + (x + y - 1)\omega}, \omega^2\frac{y(x + \omega)}{y + \omega}\right) \to (x, y),\end{aligned}$$

where $\omega^3 = 1$. Similarly we can derive recurrence equations for larger periods, but their complicated expressions are not worth to be presented here for our purpose of this paper.

The $4d$ Lotka–Volterra map $(x, y, z, u) \to (X, Y, Z, U)$ has three invariants. The invariant variety is generated by the function

$$\gamma^{(2)} = H_1 - 2 = x + y + z + u - xy - yz - zu - ux - 2$$

in the period 2 case, from which we derive the recurrence equation:

$$\begin{aligned}F_1^{(2)} &= (1 - x - z)X + x,\\ F_2^{(2)} &= Y - y(1 - x - z),\\ F_3^{(2)} &= (1 - x - z)Z + z.\end{aligned}$$

This provides an example of a three-dimensional map of period 2,

$$(x, y, z) \to \left(\frac{x}{x + z - 1}, y(1 - x - z), \frac{z}{x + z - 1}\right) \to (x, y, z). \tag{26}$$

The N-point Toda map is known equivalent to the $(d = 2N)$-dimensional Lotka–Volterra map [7]. In the case $N = 3$, the map $(x, y, z, u, v, w) \to (X, Y, Z, U, V, W)$ is defined by

$$X = y\frac{zu + zx + wu}{yw + yz + vw}, \qquad Y = z\frac{xv + xy + uv}{zu + zx + wu}, \qquad Z = x\frac{yw + yz + vw}{xv + xy + uv},$$

$$U = u\frac{yw + yz + vw}{zu + zx + wu}, \qquad V = v\frac{zu + zx + wu}{xv + xy + uv}, \qquad W = w\frac{xv + xy + uv}{yw + yz + vw}.$$

Since this map has four invariants,

$$\begin{aligned}t_1 &= x + y + z + u + v + w,\\ t_2 &= xy + yz + zx + uv + vw + wu + xv + yw + zu,\\ t_3 &= xyz,\\ t_3' &= uvw,\end{aligned}$$

the recurrence equations are expected to be four dimensional. The invariant variety of period 3 is given by the intersection of the equations [5]

$$\gamma_1^{(3)} = t_1 = 0, \qquad \gamma_2^{(3)} = t_2 = 0.$$

From these data we derive a four-dimensional recurrence equation:

$$F_1^{(3)} = (x+y+u)X + (u+v+y)y,$$
$$F_2^{(3)} = (u+v+y)Y - (u+v+y)^2 - (x-v)u,$$
$$F_3^{(3)} = (u+v+y)U + (x+y+u)u,$$
$$F_4^{(3)} = (v-x)V + (u+v+y)v,$$

or writing the solution explicitly, we find the map

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \to \begin{pmatrix} -y\dfrac{u+v+y}{x+y+u} \\ \phi(x,y,u,v) \\ -u\dfrac{x+y+u}{u+v+y} \\ v\dfrac{u+v+y}{x-v} \end{pmatrix} \to \begin{pmatrix} \phi(x,y,u,v)\dfrac{x+y+u}{x-v} \\ x\dfrac{u+v+y}{x-v} \\ u\dfrac{x-v}{u+v+y} \\ -v\dfrac{x+y+u}{x-v} \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix},$$

where

$$\phi(x,y,u,v) = \frac{(u+v+y)^2 + (x-v)u}{u+v+y}.$$

For the last example we consider the Euler top. Let (x, y, z) be the three components of the angular velocity of the Euler top. Then the map $(x, y, z) \to (X, Y, Z)$ satisfying

$$X = \alpha(Yz + Zy), \qquad Y = \beta(Zx + Xz), \qquad Z = \gamma(Xy + Yx) \tag{27}$$

defines a discrete analogue of the Euler top [8], if the parameters (α, β, γ) are related to the three moments of inertia I, J, K of the top by

$$\alpha = \frac{J-K}{2I}, \qquad \beta = \frac{K-I}{2J}, \qquad \gamma = \frac{I-J}{2K}.$$

This map has two invariants [4, 8, 9]

$$H_1 = \frac{Ix^2 + Jy^2 + Kz^2}{1-\beta\gamma x^2}, \qquad H_2 = \frac{I^2x^2 + J^2y^2 + K^2z^2}{1-\beta\gamma x^2},$$

from which we have found an invariant variety of periodic points [6]

$$v^{(3)} = \left\{ \mathbf{x} \,\middle|\, 3 + \gamma\frac{KH_1 - H_2}{IJ} - \beta\frac{JH_1 - H_2}{KI} - \left(\alpha\frac{IH_1 - H_2}{2JK}\right)^2 = 0 \right\},$$

which are generated by the function

$$\gamma^{(3)} = (1 + \beta\gamma x^2 + \gamma\alpha y^2 + \alpha\beta z^2)^2 - 4\alpha\beta\gamma(\alpha y^2 z^2 + \beta z^2 x^2 + \gamma x^2 y^2) - 4$$

in the period 3 case. The recurrence equation of period 3 is then obtained as follows:

$$\begin{aligned} F_1^{(3)} &= \beta((1-\alpha\gamma y^2)(\alpha\gamma y^2 + \beta\gamma x^2 - 2 - 2q)X - (1 - \alpha\gamma y^2 + q)x)^2 \\ &\quad - \alpha y^2(1 - \alpha\gamma y^2 + q)^2(\alpha\gamma y^2 + \beta\gamma x^2 - 1 - 2q) \\ F_2^{(3)} &= \alpha((1-\beta\gamma x^2)(\alpha\gamma y^2 + \beta\gamma x^2 - 2 - 2q)Y - (1 - \beta\gamma x^2 + q)y)^2 \\ &\quad - \beta x^2(1 - \beta\gamma x^2 + q)^2(\alpha\gamma y^2 + \beta\gamma x^2 - 1 - 2q), \end{aligned}$$

where $q = \sqrt{(1-\alpha\gamma y^2)(1-\beta\gamma x^2)}$. The map has two routes,

$$(x, y) \to \left\{ \begin{matrix} (X_+, Y_+) \to (X_-, Y_-) \\ (X_-, Y_-) \to (X_+, Y_+) \end{matrix} \right\} \to (x, y), \tag{28}$$

corresponding to the zeros of $\left\{F_1^{(3)}, F_2^{(3)}\right\}$:

$$X_\pm = \frac{1-\alpha\gamma y^2+q}{1-\alpha\gamma y^2}\frac{x \pm y\sqrt{\alpha\beta(\alpha\gamma y^2+\beta\gamma x^2-1-2q)}}{\alpha\gamma y^2+\beta\gamma x^2-2-2q},$$

$$Y_\pm = \frac{1-\beta\gamma x^2+q}{1-\beta\gamma x^2}\frac{y \pm x\sqrt{\alpha\beta(\alpha\gamma y^2+\beta\gamma x^2-1-2q)}}{\alpha\gamma y^2+\beta\gamma x^2-2-2q}.$$

The two routes correspond to the forward and the backward maps starting from the same initial point. This means that the discrete Euler top cannot start its three-period motion unless the direction of the motion is informed.

Some remarks are in order:

- Although most of the reduced maps discussed in this section happen to be rational, the solutions of (16) are not rational in general, even if the higher dimensional maps are rational. Indeed the recurrence map (28) of the discrete Euler top is not rational. We shall discuss other non-rational cases in section 4.2.
- Since an iteration of a rational map is again rational, all periodic points are rational functions of the initial point if the reduced map is rational.
- It is important to note that, among rational recurrence maps, linear maps $\mathbf{x} \to \mathbf{X} = A\mathbf{x}$ play a special role, where A is a $d \times d$ constant matrix satisfying $A^n = I$. The periodicity is rather trivial in this case. A linear map is equivalent to a set of Möbius maps by a conjugation. In fact, map (19) is a single Möbius map and (25) is a pair of two Möbius maps, which are linearizable. If we define (x_1, x_2, x_3, x_4) such that $x = x_1/x_4$, $y = x_4/x_2$, $z = x_3/x_4$, the $4d$ LV map (26) is also equivalent to the linear map

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{29}$$

  In [6] we have shown that the discrete Euler top becomes a linear map in the axially symmetric limit.
- The general solutions of the recurrence equations are difficult to express as functions of the number of the iteration. When the map is linearlizable, however, they are given by trigonometric functions. For example $X^{(n)} = ax^{\cos\pi n}$, in the case of (4), and

$$X_\pm^{(n)} = \frac{\mathrm{e}^{\pm \mathrm{i}2\pi n/3}x}{\frac{\mathrm{e}^{\pm\mathrm{i}2\pi n/3}-1}{\mathrm{e}^{\pm\mathrm{i}2\pi/3}-1}x+1},$$

  in the case of (19), corresponding to the first and the second maps.
- Once a map is linearlized the reduced system plays the role of a discrete analogue of the action–angle representation in the continuous time Hamilton systems. From this point of view it will be useful if there is a way to discriminate linearlizable maps from the rest.

4. Series of recurrence equations

Let $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{d-p}(\mathbf{x}), H_1(\mathbf{x}), \ldots, H_p(\mathbf{x})$ be some functions of $\mathbf{x} = (x_1, x_2, \ldots, x_d)$. If $g_j(\mathbf{x})$, $j = d-p+1, \ldots, d$ are the solutions of

$$H_i(f_1, f_2, \ldots, f_{d-p}, g_{d-p+1}, \ldots, g_d) = H_i(x_1, x_2, \ldots, x_d), \qquad i = 1, 2, \ldots, p,$$

they define a map

$$\mathbf{x} \to \mathbf{X} = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{d-p}(\mathbf{x}), g_{d-p+1}(\mathbf{x}), \ldots, g_d(\mathbf{x})), \tag{30}$$

in which $H_1(\mathbf{x}), H_1(\mathbf{x}), \ldots, H_p(\mathbf{x})$ are invariant.

Conversely, by using the invariants we can reduce map (30) to a $(d-p)$-dimensional one, in which the invariants become constant parameters. This fact enables us to consider the various higher dimensional maps all together, just by studying a simple low-dimensional one. Somewhat similar approaches have been developed recently in the literature [12].

4.1. Möbius map series

Following the above prescription we can derive the higher dimensional integrable maps which reduce to the Möbius map, if a, b, h are some functions of the invariants $H_1, H_2, \ldots, H_{d-1}$ and we define f_1 by

$$f_1(\mathbf{x}) = h\frac{x_1 + a}{1 + bx_1}. \tag{31}$$

The iteration of this map does not change the form of the map, but only changes the functions a, b, h. Since we have assumed that these functions are dependent on the invariants $H_i(\mathbf{x})$ alone, the initial values (a, b, h) remain constant through the iteration. If we write

$$X^{(n)} = h^{(n)}\frac{x + a^{(n)}}{1 + b^{(n)}x} \tag{32}$$

after n steps, the $(n+1)$th parameters are related to the nth ones by

$$a^{(n+1)} = \frac{a + a^{(n)}h^{(n)}}{h^{(n)} + ab^{(n)}}, \qquad b^{(n+1)} = \frac{b^{(n)} + bh^{(n)}}{1 + bh^{(n)}a^{(n)}}, \qquad h^{(n+1)} = h\frac{h^{(n)} + ab^{(n)}}{1 + bh^{(n)}a^{(n)}},$$

from which we can determine all parameters iteratively as functions of the initial values (a, b, h).

The periodicity conditions of period n for map (31) are now satisfied if the parameters (a, b, h) satisfy

$$(a^{(n+1)}, b^{(n+1)}, h^{(n+1)}) = (a, b, h). \tag{33}$$

From the construction it is clear that the periodicity conditions do not fix the values of the variable x but impose some constraints on the parameters, hence on the invariants.

Solving (33) iteratively we find the invariant varieties of periodic points [5]

$$\begin{aligned}
v^{(2)} &= \{\mathbf{x}|1 + h = 0\} \\
v^{(3)} &= \{\mathbf{x}|1 + h + h^2 + abh = 0\} \\
v^{(4)} &= \{\mathbf{x}|1 + h^2 + 2abh = 0\} \\
v^{(5)} &= \{\mathbf{x}|1 + h + h^2 + h^3 + h^4 + abh(3 + (4 + ab)h + 3h^2) = 0\} \\
v^{(6)} &= \{\mathbf{x}|1 - h + h^2 + 3abh = 0\} \\
&\;\;\vdots
\end{aligned} \tag{34}$$

According to our argument in section 3 we should have the recurrence equations corresponding to the invariant varieties (34), one for each period. To obtain the recurrence equations we must specify the invariants of the map in higher dimensions. Although the dimension of the map could be chosen arbitrary, we consider here two dimensions for the sake of simplicity. The number of the invariants is 1 in this case. Let $H(x, y)$ be the invariant. A two-dimensional map, which reduces to (31), will be obtained if we fix $f_1(x, y)$ and $H(x, y)$

as functions of (x, y). For this purpose we further assume simply that a, b are constants and the function f_1 and the invariant H are given by

$$f_1(x, y) = H(x, y)\frac{x+a}{1+bx}, \qquad H(x, y) = y(1+bx).$$

Solving $H(f_1, g) = H(x, y)$ for $g(x, y)$ we find a map

$$(x, y) \to (X, Y) = \left((x+a)y,\, y\frac{1+bx}{1+by(x+a)}\right). \tag{35}$$

This includes (17) as a special case.

Since already we have information (34) of the invariant varieties it is not difficult to derive a series of recurrence equations associated with the two-dimensional map (35), one for each period, as follows:

$$\begin{aligned}
F^{(2)} &= (1+bx)X + x + a, \\
F^{(3)} &= (1+bx)^2X^2 + (1+ab)(1+bx)(x+a)X + (x+a)^2, \\
F^{(4)} &= (1+bx)^2X^2 + 2ab(1+bx)(x+a)X + (x+a)^2, \\
F^{(5)} &= (1+bx)^4X^4 + (1+3ab)(1+bx)^3(x+a)X^3 \\
&\quad + (1+4ab+a^2b^2)(1+bx)^2(x+a)^2X^2 \\
&\quad + (1+3ab)(1+bx)(x+a)^3X + (x+a)^4, \\
F^{(6)} &= (1+bx)^2X^2 - (1-3ab)(1+bx)(x+a)X + (x+a)^2, \\
&\ \ \vdots
\end{aligned}$$

To convince ourselves let us see some of the maps explicitly. The map of period 2 is generated by $F^{(2)}$, from which we find

$$x \to -\frac{x+a}{1+bx} \to x.$$

We note that the generating functions of period 3, 4 and 6 cases are similar. There is a pair of routes for each period. The map in the period 3 case, for example, is given by

$$x \to \left\{\begin{array}{l} -\mu_+\dfrac{x+a}{1+bx} \to -\dfrac{x+a\mu_-}{\mu_-+bx} \\[2ex] -\mu_-\dfrac{x+a}{1+bx} \to -\dfrac{x+a\mu_+}{\mu_++bx} \end{array}\right\} \to x,$$

where $\mu_\pm = \frac{1}{2}\left(1+ab \pm \sqrt{(3+ab)(ab-1)}\right)$.

4.2. Biquadratic map

By studying various higher dimensional integrable maps which reduce to a one-dimensional map $x \to X$, we found, in [5], that many of them reduce not to the Möbius map but to the 'biquadratic map' $x \to X$ defined by the equation

$$aX^2x^2 + b(X+x)Xx + c(X-x)^2 + dXx + e(X+x) + f = 0. \tag{36}$$

Here,

$$\mathbf{q} = (a, b, c, d, e, f) \in \mathbf{C}^6 \tag{37}$$

are functions of $d-1$ invariants. The function $f_1(\mathbf{x})$ is determined by solving (36) for X.

Because of the symmetry of equation (36) under the exchange of X and x, the iteration of the map leaves the form of the map and changes only the parameters $\mathbf{q}$, as was shown in [5].

After repeating the iteration n times, the $(n+1)$th parameters $\mathbf{q}^{(n+1)}$ are determined by $\mathbf{q}^{(n)}$ and the initial values $\mathbf{q}$. Since the parameters are functions of the invariants alone, the periodicity conditions $\mathbf{q}^{(n+1)} = \mathbf{q}$ impose some constraints among the invariants different for each period.

Solving the periodicity conditions iteratively we have found [5] a series of invariant varieties of periodic points, one for each period. If $v^{(n)} = \{\mathbf{x}|\gamma^{(n)} = 0\}$ is the invariant variety of period n, the generating functions $\gamma^{(n)}$ are given by

$$\begin{aligned}
\gamma^{(3)}(\mathbf{q}) &= af - be - 3c^2 + cd,\\
\gamma^{(4)}(\mathbf{q}) &= 2acf - adf + b^2 f + ae^2 - 2c^3 + c^2 d - 2bce,\\
\gamma^{(5)}(\mathbf{q}) &= a^3 f^3 + (-cf^2 d + 2cfe^2 + fde^2 - 3ebf^2 - e^4 - c^2 f^2)a^2\\
&\quad + (-13c^4 f + 18c^3 fd + de^3 b + 2cf^2 b^2 + 7dc^2 e^2 - ce^2 d^2 - 2ce^3 b\\
&\quad + 2c^2 feb - 7fd^2 c^2 - 14c^3 e^2 + cd^3 f + fb^2 e^2 + f^2 db^2 - ebd^2 f)a\\
&\quad - cd^2 b^2 f - b^3 e^3 - 4c^3 deb + cdb^2 e^2 + 13ec^4 b - f^2 b^4 + 7fb^2 c^2 d\\
&\quad + c^4 d^2 - 5c^5 d + 5c^6 - 2fb^3 ec - e^2 c^2 b^2 + eb^3 df - 14fb^2 c^3
\end{aligned} \tag{38}$$

and so on.

In order to derive the recurrence equations, we must specify the higher dimensional maps. As we have shown in [5, 6] the symmetric version of the QRT map [1], the $3d$ Lotka–Volterra map of (22), the discrete Euler top, a special case of the q-Painlevé IV map belong to this category. For example the $3d$ LV map (22) is equivalent to the biquadratic map if we choose

$$\begin{aligned}
&a = r + 1, \qquad b = s - 2r - 1, \qquad c = r - s,\\
&d = s^2 + rs + 5r - 2s + 1, \qquad e = -r(s + 1), \qquad f = 0,
\end{aligned}$$

from which we could derive the invariant varieties (24).

4.3. The recurrence equations derived from the QRT map

In the rest of this section we want to derive the recurrence equations generated from the QRT map. The symmetric version of the famous QRT map [1] is given by

$$(x, y) \to (X, Y) = \left(y, \frac{\eta'(y)\rho''(y) - \rho'(y)\eta''(y) - x(\rho'(y)\xi''(y) - \xi'(y)\rho''(y))}{\rho'(y)\xi''(y) - \xi'(y)\rho''(y) - x(\xi'(y)\eta''(y) - \eta'(y)\xi''(y))}\right). \tag{39}$$

Here

$$\begin{aligned}
\xi'(x) &:= a'x^2 + b'x + c', & \xi''(x) &:= a''x^2 + b''x + c'',\\
\eta'(x) &:= b'x^2 + (d' - 2c')x + e', & \eta''(x) &:= b''x^2 + (d'' - 2c'')x + e'',\\
\rho'(x) &:= c'x^2 + e'x + f', & \rho''(x) &:= c''x^2 + e''x + f'',
\end{aligned}$$

and $\mathbf{q}' = (a', b', c', d', e', f')$ and $\mathbf{q}'' = (a'', b'', c'', d'', e'', f'')$ are constants.

Map (39) has an invariant [1]

$$H(x, y) = -\frac{\xi'(x)y^2 + \eta'(x)y + \rho'(x)}{\xi''(x)y^2 + \eta''(x)y + \rho''(x)}, \tag{40}$$

hence it can be reduced to a one-dimensional map $x \to X$. The calculation of the second elimination ideal is rather trivial in this case. If y and Y are eliminated by using the invariant $H(x, y) = h$, the result we obtain is

$$\xi(x)X^2 + \eta(x)X + \rho(x) = 0 \tag{41}$$

where

$$\xi(x) := ax^2 + bx + c, \qquad \eta(x) := bx^2 + (d - 2c)x + e, \qquad \rho(x) := cx^2 + ex + f,$$

with $\mathbf{q} = \mathbf{q}' + h\mathbf{q}''$. If we identify $\mathbf{q} = (a, b, c, d, e, f)$ with those of (37), map (41) is exactly the biquadratic map (36).

In the theory of the QRT map formula (41) is called an invariant curve [1]. The problem of solving equation (39) is now converted to finding the coefficients of (41) iteratively. Our general formula (38) provides explicit expressions of the invariant varieties of the symmetric QRT map (39). Namely, by simply replacing $\mathbf{q}$ by $\mathbf{q}' + H(x, y)\mathbf{q}''$ in (38), the invariant varieties are

$$v^{(n)} = \{x, y | \gamma^{(n)}(\mathbf{q}' + H(x, y)\mathbf{q}'') = 0\}, \qquad n = 3, 4, 5, \ldots. \tag{42}$$

In order to derive the recurrence equations we must eliminate y from the equation, so that map (39) is constrained on the variety (42). It amounts to replacing y by X in $H(x, y)$, since $X = y$. Thus we have found that the recurrence equations associated with the QRT map are generated by the functions

$$F^{(n)} = \gamma^{(n)}(\mathbf{q}' + H(x, X)\mathbf{q}''), \qquad n = 3, 4, 5, \ldots. \tag{43}$$

From (38) the recurrence equation of period 3 is, for example,

$$\begin{aligned} F^{(3)} = {} & (a' + H(x, X)a'')(f' + H(x, X)f'') - (b' + H(x, X)b'')(e' + H(x, X)e'') \\ & - 3(c' + H(x, X)c'')^2 + (c' + H(x, X)c'')(d' + H(x, X)d'') = 0. \end{aligned}$$

Note that the induced map $x \to X$ is not rational but biquadratic.

Before closing this section we would like to make a comment about the relation of our study to the geometric approach to the QRT maps [10, 11]. In the latter approach it was shown that the QRT map has periodic orbits for any initial point only when the period is 2, 3, 4, 5 or 6. Some examples are also given in [3, 4]. For example the map

$$(x, y) \to \left(y, y\frac{\alpha(x + y) - \beta xy}{\alpha(x - y) + (\beta - \gamma x)y^2} \right)$$

produces an orbit of period 4 for any initial point (x, y). This can be obtained from (39) if we set

$$d' - 2c' = e' = f' = d'' - 2c'' = e'' = f'' = 0 \tag{44}$$

and $\alpha = b'c'' - c'b''$, $\beta = c'a'' - a'c''$, $\gamma = a'b'' - b'a''$. In other words, if we choose the parameters properly the two-dimensional QRT map (39) becomes periodic for arbitrary initial points, hence for all values of the invariant $H(x, y)$. They are two-dimensional recurrence equations.

On the other hand, we have shown that the QRT map has an invariant variety (42) for each period. The formula $\gamma^{(n)}(\mathbf{q}' + H(x, y)\mathbf{q}'') = 0$ relates the invariant $H(x, y)$ to the parameters $(\mathbf{q}', \mathbf{q}'')$. Therefore, for a given set of the parameters, the value of y of the initial point (x, y) is not free but is determined dependent on x, in general. The two-dimensional recurrence equations, in which both x and y are free, are possible only for some particular set of parameters such that the value of the invariant $H(x, y)$ is irrelevant to determine the varieties (42). Indeed, if we set the parameters as (44), the equation $\gamma^{(4)}(\mathbf{q}' + H(x, y)\mathbf{q}'') = 0$ is satisfied irrespective of $H(x, y)$. The geometric approach tells us that this is possible when the period is 2, 3, 4, 5 and 6. In generic case a reduced one-dimensional map $x \to X$ is induced on the variety (42) as we replace y by X. The recurrence equation (43) associated with the QRT map holds in this way for all QRT parameters and for all periods.

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